HOW TO DRAW A STRAIGHT LINE[1]

THE great geometrician Euclid, before demonstrating to us the various propositions contained in his "Elements of Geometry," requires that we should be able to effect certain processes. These *Postulates*, as the requirements are termed, may roughly be said to demand that we should be able to describe straight lines and circles. And so great is the veneration that is paid to this master-geometrician, that there are many who would refuse the designation of "geometrical" to a demonstration which requires any other construction than can be effected by straight lines and circles. Hence many problems—such as, for example, the trisection of an angle—which can readily be effected by employing other simple means, are said to have no geometrical solution, since they cannot be solved by straight lines and circles only.

It becomes then interesting to inquire how we can effect these preliminary requirements, how we can describe these circles and these straight lines, with as much accuracy as the physical circumstances of the problems will admit of.

As regards the circle we encounter no difficulty. Taking Euclid's definition, and assuming, as of course we must, that our surface on which we wish to describe the circle is a plane, we see that we have only to make our tracing-point preserve a distance from the given centre of the circle constant and equal to the required radius. This can readily be effected by taking a flat piece of any form, such as the piece of carboard I have here, and passing a pivot which is fixed to the given surface at the given centre through a hole in the piece, and a tracer or pencil through another hole in it whose distance from the first is equal to the given radius; we shall then, by moving the pencil, be able, even with this rude apparatus, to describe a circle with considerable accuracy and ease; and when we come to employ very small holes and pivots, or even larger ones turned with all that marvellous truth which the lathe affords, we shall get a result unequalled perhaps among mechanical apparatus for the smoothness and accuracy of its movement. The apparatus I have just described is of course nothing but a simple form of a pair of compasses, and it is usual to say that the third Postulate postulates the compasses.

But the straight line, how are we going to describe that? Euclid defines it as "lying evenly between its extreme points." This does not help us much. Our text-books say that the first and second Postulates postulate a ruler. But surely that is begging the question. If we are to draw a straight line with a ruler, the ruler must itself have a straight edge; and how are we going to make the edge straight? We come back to our starting-point.

Now I wish you clearly to understand the difference between the method I just now employed for describing a circle, and the ruler method of describing a straight line. If I applied the ruler method to the description of a circle I should take a circular lamina, such as a penny, and trace my circle by passing the pencil round the edge, and I should have the same difficulty that I had with the straight-edge, for I should first have to make the lamina itself circular. But the other method I employed involves no begging the question. I do not first assume that I have a circle and then use it to trace one, but simply require that the distance between two points shall be invariable. I am of course aware that we do employ circles in our simple compass, the pivot and the hole in the moving piece which it fits are such; but they are used not because they are the curves we want to describe (they are not so, but are of a different size), as is the case with the straight-edge, but because, through the impossibility of constructing pivots or holes of no finite dimensions, we are forced to adopt the best substitute we can for making one point in the moving piece remain at the same spot. If we employ a very small pivot and hole, though they be not truly circular, the error in the description of a circle of moderate dimensions will be practically infinitesimal, not perhaps varying beyond the width of the thinnest line which the tracer can be made to describe; and even when we employ large pivots and holes we shall get results as accurate, because those pivots and holes may be made by the employment of very small ones in the machine which makes them.

It appears, then, that although we have an easy and accurate method of describing a circle, we have at first sight no corresponding means of describing a straight line; and there would seem to be a substantial difficulty in producing what mathematicians call the simplest curve, so that the question how to get over that difficulty becomes one of a decided theoretical interest.

Nor is the interest theoretical only, for the question is one of direct importance to the practical mechanician. In a large number of machines and scientific apparatus it is requisite that some point or points should move accurately in a straight line with as little friction as possible. If the ruler principle is adopted, and the point is kept in its path by guides, we have, besides the initial difficulty of making the guides truly straight, the wear and tear produced by the friction of the sliding surfaces, and the deformation produced by changes of temperature and varying strains. It becomes therefore of real consequence to obtain, if possible, some method which shall not involve these objectionable features, but possess the accuracy and ease of movement which characterises our circle-producing apparatus.

Turning to that apparatus we notice that all that is requisite to draw with accuracy a circle of any given radius is to have the distance between the pivot and the tracer properly determined, and if I pivot a second "piece" to the fixed surface at a second point having a tracer as the first piece has, by properly determining the distance between the second tracer and pivot I can describe a second circle whose radius bears any proportion I please to that of the first circle. Now, removing the tracers, let me pivot a third piece to these two *radial* pieces, as I may call them, at the points where the tracers were, and let me fix a tracer at any point on this third or *traversing* piece. You will at once see that if the radial pieces were big enough the tracer would describe circles or portions of circles on *them*, though they are in motion, with the same ease and accuracy as in the case of the simple circle drawing apparatus; the tracer will not however describe a circle on the *fixed* surface but a complicated curve.

This curve will, however, be described with all the ease and accuracy of movement with which the circles were described, and if I wish to reproduce in a second apparatus the curves which I produce with this, I have only to get the distances between the pivots and tracers accurately the same in both cases, and the curves will also be accurately the same. I could of course go on adding fresh pieces *ad libitum*, and I should get points on the structure produced, describing in general very complicated curves, but with the same results as to accuracy and smoothness, *the reproduction of any particular curve depending solely on the correct determination of a certain definite number of distances.*

These systems, built up of pieces pointed or pivoted together, and turning about pivots attached to a fixed base, so that the various points on the pieces all describe definite curves, I shall term "link-motions," the pieces being termed "links." As, however, it sometimes facilitates the consideration of the properties of these structures to regard them apart from the base to which they are pivoted, the word "linkage" is employed to denote any combination of pieces pivoted together. When such a combination is pivoted in any way to a fixed base, the motion of points on it not being necessarily confined to fixed paths, the link structure is called a "linkwork:" a "linkwork" in which the motion of every point is in some definite path being, as before stated, termed a "link-motion." I shall only add to these expressions two more: the point of a link-motion which describes any curve is called a "graph," the curve being called a "gram."

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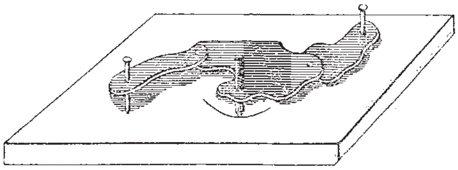
[1] Lecture at South Kensington in connection with the Loan Collection of Scientific Apparatus, by A. B. Kempe, B.A.

Fig. 1.



The consideration of the various properties of these "linkages" has occupied much attention of late years among mathematicians, and is a subject of much complexity and difficulty. With the purely mathematical side of the question I do not, however, propose to deal today, as we shall have quite enough to do if we confine our attention to the practical results which mathematicians have obtained, and which I believe only mathematicians could have obtained. That these results are valuable cannot, I think, be doubted, though it may well be that their great beauty has led some to attribute to them an importance which they do not really possess; and it may be that fifty years ago they would have had a value which, through the great improvements that modern mechanicians have effected in the production of true planes, rulers and other exact mechanical structures, cannot now be ascribed to them. But linkages have not at present, I think, been sufficiently put before the mechanician to enable us to say what value should really be set upon them.

The practical results obtained by the use of linkages are but few in number, and are closely connected with the problem of "straight-line motion," having in fact been discovered during the investigation of that problem, and I shall be naturally led to consider them if I make "straight-line motion" the backbone of my lecture. Before, however, plunging into the midst of these linkages it will be useful to know how we can practically construct such models as we require; and here is one of the great advantages of our subject—we can get our results visibly before us so very easily. Pins for fixed pivots, cards for links, string or cotton for the other pivots, and a dining-room table, or a drawing board if the former be thought objectionable, for a fixed base, are all we require. If something more artistic be preferred, the plan adopted in the models exhibited by me in the Loan Collection can be employed. The models were constructed by my brother, Mr. H. R. Kempe, in the following way. The bases are thin deal boards painted black; the links are neatly shaped out of thick cardboard (it is hard work making them, you have to sharpen your knife about every ten minutes, as the cardboard turns the edge very rapidly); the pivots are little rivets made of catgut, the heads being formed by pressing the face of a heated steel chisel on the ends of the gut after it is passed through the holes in the links; this gives a very firm and smoothly working joint. More durable links may be made of tin-plate; the pivot-holes must in this case be punched, and the eyelets used by bootmakers for laced boots employed as pivots; you can get the proper tools at a trifling expense at any large tool shop.

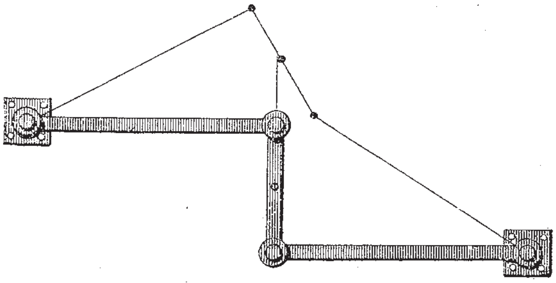


Fig. 2.

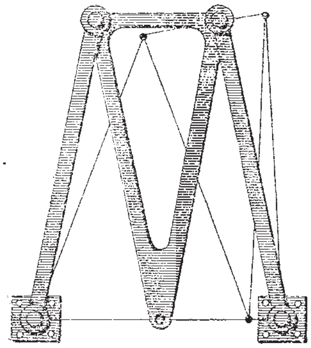


Fig. 3.

Now, as I have said, the curves described by the various points on these link-motions are in general very complex. But they are not necessarily so. By properly choosing the distances at our disposal we can make them very simple. But can we go to the fullest extent of simplicity and get a point on one of them moving accurately in a straight line? That is what we are going to investigate.

To solve the problem with our single link is clearly impossible: all the points on it describe circles. We must therefore go to the next simple case—our three-link motion. In this case you will see that we have at our disposal the distance between the fixed pivots, the distances between the pivots on the radial links, the distance between the pivots on the traversing link, and the distances of the tracer from those pivots; in all six different distances. Can we choose those distances so that our tracing-point shall move in a straight line?

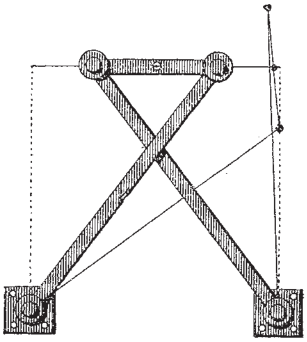


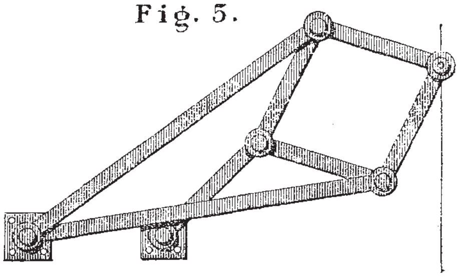
Fig. 4.

The first person who investigated this was that great man James Watt. "Watt's Parallel Motion," invented in 1784, is well known to every engineer, and is employed in nearly every beam-engine. The apparatus reduced to its simplest form is shown in Fig. 2.

The radial bars are of equal length—I employ the word "length" for brevity, to denote the distance between the pivots, the links, of course, may be of any length or shape,—and the distance between the pivots or the traversing link is such that when the radial bars are parallel the line joining those pivots is perpendicular to the radial bars. The tracing-point is situate half-way between the pivots on the traversing piece. The curve described by the tracer is, if the apparatus does not deviate much from its mean position, approximately a straight line. The reason of this is that the circles described by the extremities of the radial bars have their concavities turned in opposite directions, and the tracer being half-way between, describes a curve which is concave neither one way nor the other, and is therefore a straight line. The curve is not, however, accurately straight, for if I allow the tracer to describe the whole path it is capable of describing, it will, when it gets some distance from its mean position, deviate considerably from the straight line, and will be found to describe a figure 8, the portions at the crossing being nearly straight. We know that they are not quite straight, because it is impossible to have such a curve partly straight and partly curved.

For many purposes the straight line described by Watt's apparatus is sufficiently accurate, but if we require an exact one it will, of course, not do, and we must try again. Now it is capable of proof that it is impossible to solve the problem with three moving links; closer approximations to the truth than that given by Watt can be obtained, but still not actual truth.

Fig. 5.



I have here some examples of these closer approximations. The first of these, shown in Fig. 3, is due to Richard Roberts of Manchester.

The radial bars are of equal length, the distance between the fixed pivots is twice that of the pivots on the traversing piece, and the tracer is situate on the traversing piece, at a distance from the pivots on it equal to the lengths of the radial bars. The tracer in consequence coincides with the straight line joining the fixed pivots at those pivots and half-way between them. It does not, however, coincide at any other point, but deviates very slightly between the fixed pivots. The path described by the tracer when it passes the pivots, altogether deviates from the straight line.

The other apparatus was invented by Prof. Tchebicheff of St. Petersburg. It is shown in Fig. 4. The radial bars are equal in length, being each in my little model five inches long. The distance between the fixed pivots must then be four inches, and the distance between the pivots or the traversing bar two inches. The tracer is taken half-way between these last. If now we draw a straight line—I had forgotten that we cannot do that yet, well, if we draw a straight line, popularly so called—through the tracer in its mean position as shown in the figure, parallel to that forming the fixed pivots, it will be found that the tracer will coincide with that line at the points where verticals through the fixed pivots cut it as well as at the mean position, but, as in the case of Roberts's parallel motion, it coincides nowhere else, though its deviation is very small as long as it remains between the verticals.

We have failed then with three links, and we must go on to the next case, a five-link motion—for you will observe that we must have an odd number of links if we want an apparatus describing definite curves. Can we solve the problem with five? Well, we can, but this was not the first accurate parallel motion discovered, and we must give the first inventor his due (although he did not find the simplest way), and proceed in strict chronological order.

In 1864, eighty years after Watt's discovery, the problem was first solved by M. Peaucellier, an officer of Engineers in the French army. His discovery was not at first estimated at its true value, fell almost into oblivion, and was rediscovered by a Russian student named Lipkin, who got a substantial reward from the Russian Government for his supposed originality. However, M. Peaucellier's merit has at last been recognised, and he has been awarded the great mechanical prize of the Institute of France, the "Prix Montyon."

M. Peaucellier's apparatus is shown in Fig. 5. It has, as you see, seven pieces or links. There are first of all two long links of equal length. These are both pivoted at the same fixed point; their other extremities are pivoted to opposite angles of a rhombus composed of four equal shorter links. The portion of the apparatus I have thus far described, considered apart from the fixed base, is a linkage termed a "Peaucellier cell." We then take an *extra* link, and pivot it to a fixed point whose distance from the first fixed point, that to which the cell is pivoted, is the same as the length of the extra link; the other end of the extra link is then pivoted to one of the free angles of the rhombus; the other free angle of the rhombus has a pencil at its pivot. That pencil will accurately describe a straight line.

I must now indulge in a little simple geometry. It is absolutely necessary that I should do so in order that you may understand the principle of our apparatus.

(*To be continued.*)

FOSSIL FLORAS AND GLACIAL PERIODS

A RECENT notice in NATURE (vol. xiv. p. 336) of certain inferences of Prof. Heer in connection with the Arctic fossil plants obtained by the Swedish Expeditions of 1870 and 1872, suggests some thoughts on the relations of fossil plants to climate, which, though I have discussed them elsewhere, deserve to have attention again directed to them. In my Bakerian Lecture before the Royal Society in 1870, and in my "Report on the Pre carboniferous Flora of Canada," published by the Canadian Survey in 1871, I deduced from the generalisations of Prof. James Hall as to the growth of the American Continent from the north-east, in connection with the distribution of the fossil plants of the Upper Silurian, Erian, and Carboniferous systems, the conclusion that these assemblages of plants entered North America from the north-east, and propagated themselves southward and westward. Prof. Asa Gray had, as early as 1867, stated similar conclusions with reference to the modern floras of America and Eastern Asia, and has more recently extended them to the Tertiary floras on the evidence of Heer and Lesquereux.[1]

The further conclusion that all the old floras appeared suddenly and abruptly in the temperate regions, and with a great number of species, I have illustrated in the Report above referred to, as far as regards the Palæozoic plants, and have referred to the evidence of it in the case of the Cretaceous and Tertiary floras in my address to the American Association in 1875.

With regard to the succession of these floras, it is true that it has been the fashion with certain European palæontologists to regard our rich Devonian or Erian flora

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[1] Address to the American Association, 1872.